APPROACH TO ARTINIAN ALGEBRAS VIA NATURAL QUIVERS

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Abstract. Given an Artinian algebra A over a field k, there are several combinatorial objects associated to A. They are the diagram D_A as defined in [DK], the natural quiver Δ_A defined in [Li] (cf. Section 2), and a generalized version of k-species $(A/r, r/r^2)$ with r being the Jacobson radical of A. When A is splitting over the field k, the diagram D_A and the well-known ext-quiver Γ_A are the same. The main objective of this paper is to investigate the relations among these combinatorial objects and in turn to use these relations to give a characterization of the algebra A.

1. Introduction

1.1. Given an Artinian algebra A over a field k, there are several combinatorial objects associated to A. They are the diagram D_A as defined in [DK], the natural quiver Δ_A defined in [Li] (cf. Section 2), and a generalized version of k-species $(A/r, r/r^2)$ with r being the Jacobson radical of A. When A is splitting over the field k, the diagram D_A and the well-known ext-quiver Γ_A are the same. The main objective of this paper is to investigate the relations among these combinatorial objects and in turn to use these relations to give a characterization of the algebra A.

For a given Artinian k-algebra A, let $\{S_1, S_2, \cdots, S_s\}$ be the complete set of non-isomorphic irreducible A-modules. Set $D_i = \mathrm{End}_A(S_i)$ which is a division ring and $\mathrm{Ext}^1_A(S_i, S_j)$ is a D_i-D_j-bimodule. A is said to *split* over the ground field k, or say, to *be k-splitting*, if $\dim_k \mathrm{End}_A(S_i) = 1$ (i.e., $D_i = k$ for all irreducible A-modules S_i. Recall that a quiver is a finite directed graph $\Gamma = (\Gamma_0, \Gamma_1)$ with vertex set Γ_0 and arrow set Γ_1. For a k-splitting algebra A, one can define a finite quiver Γ_A called the *Ext-quiver* of A by setting $\Gamma_0 = \{1, 2, \cdots, s\}$, and $m_{ij} = \dim_k(\mathrm{Ext}^1_A(S_i, S_j))$ being the number of arrows from i to j.

There is another to characterize the Ext-quiver for a k-splitting algebra Artinian algebra. By [ARS] and [Liu], when A is a finite-dimensional algebra over a field k and $1 = \varepsilon_1 + \cdots + \varepsilon_s$ a decomposition of 1 into a sum of primitive orthogonal idempotents, then, we can re-index $\{S_1, S_2, \cdots, S_s\}$ such that

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$S_i \cong A\varepsilon_i / r\varepsilon_i$ where r is the radical of A, and moreover, $\dim_k \mathrm{Ext}_A(S_i, S_j) = \dim_k(\varepsilon_j r / r^2 \varepsilon_i)$.

If A is basis and k-splitting, then A is a quotient of the path algebra of Γ_A. The properties of this quiver Γ_A, in particular, its relation to the representations of A has been extensively studied in the field of representations of algebras. Γ_A is invariant under Morita equivalence, i.e., if A and B are two Morita equivalent k-algebras, then Γ_A is isomorphic to Γ_B.

If A is not basic, it is not longer isomorphic to a quotient of the path algebra $k\Gamma_A$. It is discussed in this paper and [Li] how to get the analogue of the Gabriel theorem in this case.

1.2. In recent years, geometric methods has been heavily used in representation theory of algebras. To each finite dimensional algebra A over an algebraically closed field k, one can associate a sequence of algebraic varieties $\mathrm{Mod}_k^d(A)$ $(d = 1, 2, 3 \ldots,)$ as closed subvarieties of the affine spaces $k^{d \times d}$. The association of the varieties depends on the presentation of the algebra A using finitely many generators and finitely many relations. In [B], it is proved that two algebras A and B are isomorphic if and only if the associated varieties $\mathrm{Mod}_k^d(A)$ and $\mathrm{Mod}_k^d(B)$ are isomorphic as $GL_d(k)$-varieties. Thus having a more standard presentation of the algebra A will help with studying these varieties. The purpose of these paper is to explore relations of the Ext-quiver of A and the natural quiver which will be defined in Section 2. The natural quiver will have fewer arrows than the Ext-quiver when the algebra A is not basic. Natural quivers are not invariant under the Morita equivalence and much closer to reflect the structure of the algebra, rather than just its module category. There are numerous cases even in the representation theory that one needs the structure of the algebras, for example, the character values of finite groups in a block cannot be preserved through Morita equivalence.

1.3. The paper is organized as follows. In Section 2 we recall the definition of the natural quiver Δ_A of an Artinian k-algebra A and provide a precise relation with the Ext-quiver Γ_A when the algebra A is splitting over the ground field k. In Section 3, we prove in Theorem 3.4 that any Artinian algebra, which is splitting over its radical, is a quotient of the generalized path algebra of its natural quiver associated to A/r. This gives a presentation of the algebra A. Although there is always a surjective algebra homomorphism from the path algebra of the natural quiver Δ_A to the tensor algebra $T_{A/r}(r/r^2)$, the above surjective map to A does not always factor through $T_{A/r}(r/r^2)$ (Example 3.5). There have been numerous generalizations of Wedderburn-Malcev theorems to characterize an Artinian algebra that is splitting over its radical. By using the generalized path algebra of the natural quiver, we give another characterization of an Artinian algebra A which is splitting over its radical, see Corollary 3.6. Moreover, we discuss the relations among the natural quiver and the Ext-quiver of an Artinian

algebra and the associated generalized path algebra, see the figure in the end of Section 4, and furthermore, their relationship with the diagram of an Artinian algebra as defined in [DK]. The main results are the formulae in Theorems 2.2 and 5.3. As an application, in Section 5, we discuss the relationship between the diagram and natural quiver of an artinian algebra that is a not splitting over the ground field. In Section 6 we prove that a (not necessarily basic) hereditary Artinian algebra which is splitting over its radical is isomorphic to the generalized path algebras of its natural quiver provided the defining ideal as described in Theorem 3.4 does not interest with the arrow space (see Theorem 6.5).

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2. The relation between natural quiver and Ext-quiver

Suppose that A is a left Artinian k-algebra, and $r = r(A)$ is its Jacobson radical.

Write $A/r = \bigoplus_{i=1}^{s} A_i$ where A_i are two-sided simple ideals of A/r. Such decomposition of A/r is also called block decomposition of the algebra A/r. Then, r/r^2 is an A/r-bimodule. Let ${}_iM_j = A_i \cdot r/r^2 \cdot A_j$ which is finitely generated as A_i-A_j-bimodule for each pair (i, j).

For two rings A and B, and a finitely generated A-B-bimodule M, define $\mathrm{rk}_{A,B}(M)$ to be the minimal number of generators of M as a A-B-bimodule among all generating sets. As a convention, we always denote $\mathrm{rk}_{A,B}(0) = 0$.

The isomorphism classes of irreducible A-modules is indexed by the set $\Delta_0 = \{1, \cdots, s\}$ corresponding to the set of blocks of A/r. We now define the natural quiver $\Delta_A = (\Delta_0, \Delta_1)$ with Δ_0 being the vertex set and, for $i, j \in \Delta_0$, $t_{ij} = \mathrm{rk}_{A_j,A_i}({}_jM_i)$ being the number of arrows from i to j in Δ_A. Obviously, there is no arrow from i to j if ${}_jM_i = 0$. This quiver $\Delta_A = (\Delta_0, \Delta_1)$ is called the *natural quiver* of A.

The notion of natural quiver was first introduced in [Li], where the aim was to use the generalized path algebra from the natural quiver of an Artinian algebra A to characterize A through the generalized Gabriel theorem. The advantage of generalized path algebra is that valued quiver information is already encoded in the generalized path algebras. In the language of Kontsevich and Soilbelman [KY], Gabriel type algebra cannot be stated as an affine non-commutative scheme which can be embedded into a thin scheme in a sense that they are "infinitesimally" isomorphic. Result of this paper will be to find a "smallest" embedding.

For a quiver $Q = (Q_0, Q_1)$, a sub-quiver Q' of Q is called *dense* if $(Q')_0 = Q_0$ and for any vertices i, j, there exist an arrow from i to j in Q' if and only if there exist an arrow from i to j in Q.

When A is splitting over the ground field k, then $A_i \cong M_{n_i}(k)$ and the irreducible module S_i has k-dimension n_i. In this case the Ext-quiver Γ_A of

A is defined (cf. 1.1). It is proved in [Liu, Prop. 7.4.3] that

$$t_{ij} \leq m_{ij} \leq n_i n_j t_{ij}.$$

In addition, if A is basic, then $\Delta_A = \Gamma_A$ as discussed in the 1.1 In general these two constructions will give two different quivers if A is not basic. The following results were proved in [LC].

Proposition 2.1. *Let A be a k-splitting Artinian k-algebra over a field k and B is the corresponding basic algebra of A. Then,*

(i) $\Gamma_A = \Gamma_B$*;*
(ii) $\Gamma_B = \Delta_B$*;*
(iii) Δ_A *is a dense sub-quiver of* Γ_A*, and thus a dense sub-quiver of* Γ_B *and* Δ_B*.*

Now, we will just give the exact relation between t_{ij} and m_{ij} when the algebra A is splitting over k. First, we recall that for any real number a, the ceiling of a is defined to be

$$\lceil a \rceil = \min\{n \in \mathbb{Z} \mid n \geq a\}.$$

Theorem 2.2. *Let A be an Artinian k-algebra A which is splitting over k. Assume Γ_A is the Ext quiver of A and Δ_A is the natural quiver of A. Then $t_{ij} = \lceil \frac{m_{ij}}{n_i n_j} \rceil$ for $i, j \in (\Gamma_A)_0 = (\Delta_A)_0$. Here m_{ij} is the number of arrows from i to j in Γ_A.*

Proof. The proof involves computing a minimal generating set of ${}_iM_j = A_i \cdot (r/r^2) \cdot A_j$ as A_i-A_j-bimodules. We first note that both A_i and A_j are simple k-algebras and split over k. Hence $A_i \otimes_k A_j^{op}$ is a also a simple central k-algebra, isomorphic to $M_{n_i n_j}(k)$. Since r/r^2 is a semisimple left A_i-module and a semisimple right A_j-module, ${}_iM_j$ is a semisimple $A_i \otimes_k A_j^{op}$-module with simple components isomorphic to $S_i \otimes_k S_j^{op}$. Here S_j^{op} is the right irreducible A_j-module. Let e_{kl}^i be the standard matrix basis elements of $A_i = M_{n_i}(k)$. Then $\dim_k \mathrm{Ext}_A^1(S_i, S_j) = \dim_k e_{11}^i r/r^2 e_{11}^j$. We claim that ${}_iM_j$ has exactly $\dim_k e_{11}^i r/r^2 e_{11}^j$ many $A_i \otimes_k A_j^{op}$-composition factors. Indeed $e_{11}^i \otimes e_{11}^j$ is a primitive idempotent of $A_i \otimes_k A_j^{op}$. Now the theorem follows from the following lemma for simple Artinian rings. □

Lemma 2.3. *Let $R = M_n(D)$ be the ring of all $n \times n$-matrices with entries in a division ring D and M be an R-module. Let $L = D^n$ be the natural irreducible R-module of column vectors. Then*

(i) *M is a semisimple R-module isomorphic to $L^{\oplus m} = L \oplus L \oplus \cdots \oplus L$, where $m = \dim_D(e_{11}M)$;*
(ii) *M can be generated by $\lceil \frac{m}{n} \rceil$ many elements over R, but cannot be generated by fewer number of elements.*

Proof. Since R is simple, all finitely generated R-modules are semisimple. Note that every irreducible R-module is isomorphic to $L = D^n$. Then

(i) follows from the fact that $\dim_D e_{11}L = 1$. To show (ii), we note that $R = \oplus_{s=1}^{n} Re_{ss} \cong L^{\oplus n}$ as left R-module. If M is generated by t-many elements, then M is a quotient of $R^{\oplus t}$ as a left R-module. Hence, M has at most tn composition factors counting multiplicity, i.e., $m \leq tn$ and $\lceil \frac{m}{n} \rceil \leq t$. Thus M cannot be generated by less than $\lceil \frac{m}{n} \rceil$ many elements. On the other hand, let N be any R-module of length $p \leq n$. Then N is isomorphic to $L^{\oplus p}$. There is a surjective homomorphism $\phi : R \to N$ of left R-modules. By writing $m = tn + s$, with $0 \leq s < n$, we have $\lceil \frac{m}{n} \rceil = t$ if $s = 0$ and $\lceil \frac{m}{n} \rceil = t + 1$ if $s > 0$. Hence we can construct a surjective homomorphism $R^{\lceil \frac{m}{n} \rceil} \to L^{\oplus tn+s} = (L^{\oplus n})^{\oplus t} \oplus L^{\oplus s}$. Hence M is generated by $\lceil \frac{m}{n} \rceil$ many elements. $\square$

As we note in Proposition 2.1 (ii), the Ext-quiver and the natural quiver of a finite dimensional basic algebra coincide each other. As an application of Theorem 2.2, we give an example which means the coincidence is also possible to happen for some non-basic algebras.

Example 2.4. Let k be a field of characteristic different from 2 and let Q be the quiver:

$$\begin{tikzcd} e_3 & e_2 \arrow[l, "\beta"'] & e_1 \arrow[l, "\alpha"'] \arrow[r, "\alpha'"] & e_{2'} \arrow[r, "\beta'"] & e_{3'} \end{tikzcd}$$

Let $\Lambda = kQ$ be the path algebra of Q and $G = \langle \sigma \rangle$ be the automorphism group of Q of order 2. Then σ defines a k-algebra automorphism of kQ. Now, we consider the Ext-quiver and the natural quiver of the skew group algebra ΛG (see [ARS]).

Let r be the Jacobson radical of Λ. By Proposition 4.11 in [ARS], $r\Lambda G)$ is the Jacobson radical of ΛG. It is easy to see that $(\Lambda G)/(r\Lambda G) \cong (\Lambda/r)G$. In Page 84 of [ARS], it was given that $(\Lambda/r)G \cong A_1 \times A_2 \times A_3 \times A_4 = k \times k \times \begin{pmatrix} k & k \\ k & k \end{pmatrix} \times \begin{pmatrix} k & k \\ k & k \end{pmatrix}$ as algebras and the associated basic algebra B is obtained in the reduced form from ΛG, which is Mortia-equivalent to ΛG, and moreover, it was proved in [ARS] that B is isomorphic to the path algebra of the following quiver

$$\begin{tikzcd} & e_{(3)} \arrow[d, "\nu"] & \\ e_{(2)} \arrow[r, "\lambda"] & e_{(1)} \arrow[r, "\mu"] & e_{(4)} \end{tikzcd}$$

This quiver is just the Ext-quiver $\Gamma_{\Lambda G}$ of ΛG. Therefore, all $m_{ij} = 0$, or 1. For $i = 1, 2, 3, 4$, $\dim_k A_i = n_i^2$ where $n_1 = n_2 = 1$, $n_3 = n_4 = 2$. By Theorem 2.2, $t_{ij} = \lceil \frac{m_{ij}}{n_i n_j} \rceil$. Then, for each pair (i, j), we have $t_{ij} = m_{ij} = 0$ or 1. Therefore, the natural quiver $\Delta_{\Lambda G}$ is equal to the Ext-quiver $\Gamma_{\Lambda G}$.

3. Algebras splitting over radicals

The concept of generalized path algebra was introduced early in [CL]. Here we review a different but equivalent definition.

Given a quiver $Q = (Q_0, Q_1)$ and a collection of k-algebras $\mathcal{A} = \{A_i \mid i \in Q_0\}$, let $e_i \in A_i$ be the identity. Let $A_0 = \prod_{i \in Q_0} A_i$ be the direct product k-algebra. Note that e_i are orthogonal central idempotents of A_0.

For $i, j \in Q_0$, let $\Omega(i,j)$ be the subset of arrows in Q_1 from i to j. Define $A_i\Omega(i,j)A_j$ to be the free A_i-A_j-bimodule (in the category of k-vector spaces) with basis $\Omega(i,j)$. This is the free $A_i \otimes_k A_j^{op}$-module over the set $\Omega(i,j)$.

Then $M = \oplus_{i,j} A_i\Omega(i,j)A_j$ is an A_0-A_0-bimodule. The generalized path algebra is defined to be the tensor algebra

$$T_{A_0}(M) = \oplus_{n=0}^{\infty} M^{\otimes_{A_0} n}. \tag{1}$$

Here $M^{\otimes_{A_0} n} = M \otimes_{A_0} M \otimes_{A_0} \cdots \otimes_{A_0} M$ and $M^{\otimes_{A_0} 0} = A_0$. We denote the generalized path algebra by $k(Q, \mathcal{A})$. Elements in $M^{\otimes_{A_0} n}$ are called *virtual $\mathcal{A}$-paths* of length n. In cases of path algebras, virtual $\mathcal{A}$-paths are linear combinations of $\mathcal{A}$-paths of equal length. We denote $J = \bigoplus_{n=1}^{+\infty} M^{\otimes_{A_0} n}$. Then $k(Q, \mathcal{A})/J \cong A_0$.

The generalized path algebra has the following universal mapping property. For any k-algebra B with any k-algebra homomorphism $\phi_0 : A_0 \to B$ (thus making B an A_0-A_0-bimodule) and any A_0-A_0-bimodule homomorphism $\phi_1 : M \to B$, there is a unique k-algebra homomorphism $\phi : k(Q, \mathcal{A}) \to B$ extending ϕ_0 and ϕ_1.

This definition is equivalent to the original definition in [CL] and has the advantage of the above mentioned universal mapping property. As a matter of fact, the classical path algebras are the special cases by taking $A_i = k$. We are more interested in the case when A_i are simple k-algebras, in particular when all A_i are central simple algebras. The generalized path algebra $k(Q, \mathcal{A})$ is called *normal* if all A_i are simple k-algebras.

A k-algebra A is said to be *splitting over its radical* r if there is a k-algebra homomorphism $\rho : A/r \to A$ such that $\pi \circ \rho = \mathrm{Id}_{A/r}$. For example if A/r is separable and A is Artinian, then A is always splitting over r as result of Wedderburn-Malcev theorem (and many generalizations in the literature [P]). Note that a normal generalized path algebra $k(Q, \mathcal{A})$ with an acyclic (no oriented cycles) quiver Q is always splitting over its radical $\mathrm{rad}(k(Q, \mathcal{A})) = J$. This equality fails in general for normal generalized path algebras.

Proposition 3.1. *Let Q be a finite quiver and $k(Q, \mathcal{A})$ be a normal generalized path algebra with each A_i being finite dimensional. If I is an ideal of $k(Q, \mathcal{A}$ with $J^s \subseteq I \subset J$ for some positive integer s. Then, $k(Q, \mathcal{A})/I$ is a finite dimensional algebra and* $\mathrm{rad}(k(Q, \mathcal{A})/I) = J/I$.

Proof. Since Q is finite quiver and A_i are finite dimensional, we have $M^{\otimes_{A_0} n}$ in (1) is finite dimensional over k for all n. Hence $k(Q,\mathcal{A})/J^s$ is also finite dimensional. It follows that $k(Q,\mathcal{A})/I$ is finite dimensional. On the other hand, J/I is a nilpotent ideal of $k(Q,\mathcal{A})/I$ with $(J/I)^s = 0$ and $(k(Q,\mathcal{A})/I)/(J/I) \cong k(Q,\mathcal{A})/J \cong \bigoplus_{i\in Q_0} A_i$ is a semisimple algebra. This implies $\mathrm{rad}(k(Q,\mathcal{A})/I) = J/I$. □

We remark that the finite dimensionality of A_i over k cannot be removed. For example, take the Dynkin quiver Q of type A_2 with two vertices $\{1,2\}$ and one arrow α from 1 to 2. Let A_1 and A_2 be two infinite dimensional field extensions of k. Although A_1 and A_2 are two simple k-algebras, but the path algebra $k(Q,\mathcal{A})$ is not left or right Artinian since

$$A_2\alpha V = \{\sum_i a_i\alpha b_i : a_i \in A_2, b_i \in V\}$$

is a left ideal of $k(Q,\mathcal{A})$ for any vector subspace V of A_1.

In this section we will show that every Artinian algebra A which is splitting over its radical will be a quotient of a generalized path algebra of its natural quiver.

For an Artinian algebra A with radical r, let Δ_A be its natural quiver and $A/r = \bigoplus_{i\in(\Delta_A)_0} A_i$ where all A_i are simple algebras. Denote $\mathcal{A} = \{A_i : i \in \Delta_0\}$. Then, the generalized path algebra $k(\Delta_A,\mathcal{A})$ is called *the associated generalized path algebra* of A.

In [LC], we introduce the following notion of Gabriel-type algebra.

Definition 3.2. Let A be an Artinian k-algebra and $k(\Delta_A,\mathcal{A})$ be its associated generalized path algebra. A is said to be of *Gabriel-type for generalized path algebra* if $A \cong k(\Delta_A,\mathcal{A})/I$ for some ideal I of $k(\Delta_A,\mathcal{A})$ contained in J. We call I the defining ideal of A.

In [Li] and [LC], the conditions for certain artinian algebras to be Gabrial type were discussed under different assumptions. Here we give a more general description.

Theorem 3.3. *Let A be a Gabriel-type Artinian algebra for generalized path algebra over a field k such that $A \cong k(\Delta_A,\mathcal{A})/I$ with the ideal I of $k(\Delta_A,\mathcal{A})$ satisfying $I \subset J$. Then, A is splitting over its radical r.*

Proof. The assumption $I \subseteq J$ implies

$$(k(\Delta_A,\mathcal{A})/I)/(J/I) \cong k(\Delta_A,\mathcal{A})/J \cong A_1 \oplus \cdots \oplus A_p \cong A/r$$

is semisimple, then $\mathrm{rad}\, A \cong \mathrm{rad}(k(\Delta_A,\mathcal{A})/I) = J/I$. We have $A_1 \oplus \cdots \oplus A_p \subset k(\Delta_A,\mathcal{A})$ and $(A_1 \oplus \cdots \oplus A_p) \cap I \subset (A_1 \oplus \cdots \oplus A_p) \cap J = 0$. Hence, we get $A_1 \oplus \cdots \oplus A_p \overset{\varepsilon}{\hookrightarrow} k(\Delta_A,\mathcal{A})/I \overset{\eta}{\twoheadrightarrow} (k(\Delta_A,\mathcal{A})/I)/(J/I) \cong k(\Delta_A,\mathcal{A})/J \cong A_1 \oplus \cdots \oplus A_p$ which implies $\eta\varepsilon = 1$. Therefore, A is splitting over r. □

In fact, in Theorem 8.5.4 of [DK], it was proven that, for a finite dimensional algebra A with radical r, if the quotient algebra A/r is separable,

then A is isomorphic to a quotient algebra of the tensor algebra $T_{A/r}(r/r^2)$ by an ideal I such that $J^s \subset I \subset J^2$ for some positive integer s. The separability condition plays to two roles in the proof. First, it guarantees the Wedderburn-Malcev theorem to get

(a) A is splitting over its radical.

Secondly, the seperability also implies that

(b) r/r^2 isomorphic to a direct summand of r as an A_0-A_0-bimodule.

It turns out the properties (a) and (b) together is equivalent to the existence of the surjective map ϕ in the theorem.

It is proved in [Li] that there always exists a surjective homomorphism of algebras $\pi : k(\Delta_A, \mathcal{A}) \to T_{A/r}(r/r^2)$. This can be seen from the universal property of $k(\Delta_A, \mathcal{A})$. Hence any artinian algebra A with separable quotient A/r is isomorphic to a quotient algebra of $k(\Delta_A, \mathcal{A})$ by an ideal I as in Theorem 3.3. The different point in this paper and that in [Li] the algebra A is splitting over its radical and the given ideal I is included in J but not in J^2. The following theorem is an improvement of the result in [Li].

Theorem 3.4. *Let A be an Artinian k-algebra such that A is splitting over its radical. Then there is a surjective algebra homomorphism $\phi : k(\Delta_A, \mathcal{A}) \to A$ with $J^s \subseteq \ker(\phi) \subseteq J$ for some positive integer s.*

Proof. Let $A/r = A_0 = \oplus_{i\in\Delta_0} A_i$. Then A_0 is semisimple and r/r^2 is a semisimple A_0-module. Since A splits over r, we can regard A_0 as a subalgebra of A and hence r is an A_0-A_0-bimodule under the multiplication in A and the quotient map $^-: r \to r/r^2$ is a homomorphism of A_0-A_0-bimodules. We will used this property in the following argument.

For each pair (i, j), let ${}_iM_j = A_i(r/r^2)A_j$. Let $X_{ij} = \{f_l^{ij} \in A_i r A_j \subseteq r \mid l = 1, \dots, t_{ij}\}$ such that the image $\bar{X}_{ij} = \{\bar{f}_l^{ij} \mid l = 1, \dots, t_{ij}\}$ in ${}_iM_j$ is a minimal generating set as an A_i-A_j-bimodule. Let $X = \cup_{i,j\in\Delta_0} X_{ij}$. Then, its image $\bar{X} = \cup_{i,j\in\Delta_0} \bar{X}_{ij}$ generates $r/r^2 = \sum_{i,j\in\Delta_0} A_i(r/r^2)A_j$ as an A_0-A_0-bimodule.

We now show that the subalgebra S of A generated by A_0 and the set X is actually A. Consider the associated graded algebra $\mathrm{gr}(A) = \oplus_{s\geq 0} r^s/r^{s+1}$, with $r^0 = A$. Then $\mathrm{gr}(S) = \oplus_{s\geq 0}(S\cap r^s)/(S\cap r^{s+1})$ is the graded subalgebra of $\mathrm{gr}(A)$ generated by A_0 and $\bar{X} = \{\bar{f}_l^{ij} \mid 1 \leq l \leq t_{ij}, i, j \in \Delta_0\}$.

We claim that $\mathrm{gr}(S) = \mathrm{gr}(A)$. We will show that $S_p = r^p/r^{p+1}$, where $S_p = (S \cap r^p)/(S \cap r^{p+1})$ are the homogeneous components of the graded algebras $\mathrm{gr}\, S$. Note that both S_p and r^p/r^{p+1} are A_0-A_0-bimodules. It follows from the definition that $S_1 = (A_0XA_0)/(r^2 \cap S) \subseteq A_0\bar{X}A_0 = r/r^2$. By the choice of the set $\bar{X}$, we have A_0XA_0 maps onto r/r^2 (under the quotient map $r \to r/r^2$). Hence, we have $S_1 = r/r^2$. The multiplication

$$\underbrace{r/r^2 \otimes_{A_0} r/r^2 \otimes_{A_0} \cdots \otimes_{A_0} r/r^2}_{p \text{ times}} \to r^p/r^{p+1}$$

in $\mathrm{gr}(A)$ is surjective following the definition of r^p. Thus

$$r^p/r^{p+1} = \underbrace{S_1 \cdots S_1}_{p \text{ times}} \subseteq S_p.$$

But we have $S_p \subseteq r^p/r^{p+1}$. Hence $S_p = r^p/r^{p+1}$ and $\mathrm{gr}(A) = \mathrm{gr}(S)$.

We now prove that $S = A$. Otherwise, we must have $r \nsubseteq S$ since A is generated by A_0 and r. Let p be maximal such that $r^p \subseteq S$ is not true. Such p exists since $r^m = 0$ for $m >> 0$ (A is Artinian) and $p \geq 1$. Take $a \in r^p \setminus S$ and let $\bar{a} \in r^p/r^{p+1}$ be the image of a. Since $S_p = r^p/r^{p+1}$, there is $b \in S \cap r^p$ such that $\bar{b} = \bar{a}$ in r^p/r^{p+1}. Thus $a - b \in r^{p+1} \subseteq S$ (p is maximal). Hence, $a = b + (a - b) \in S \cap r^p$. This is a contradiction. Hence we have $S = A$.

Since A splits over r and A_0 is a subalgebra of A such that the quotient map $A \to A/r$ restricts to A_0 is the identity map. Then $A_0XA_0 \subseteq A$ is an A_0-A_0-sub-bimodule of A. Since $|\Omega(i,j)| = t_{ij} = rk({}_iM_j)$, there is a surjective homomorphism of A_0-A_0-bimodules: $A_0\Omega(i,j)A_0 \to A_0X_{ij}A_0$. Now by the universal mapping property of the generalized path algebra, there is a surjective algebra homomorphism $\phi : k(\Delta_A, \mathcal{A}) \to S$ since S is generated by A_0 and the set X as a subalgebra of A. Hence $\phi(k(\Delta, \mathcal{A})) = S = A$ as we have just proved and $\phi\mid_{A_0} = \mathrm{id}_{A_0}$.

Let $I = \ker(\phi)$ and J be the ideal of $k(\Delta_A, \mathcal{A})$ generated by all virtual $\mathcal{A}$-paths of length 1. Then $\phi(J) = r$ and thus induces $k(\Delta, \mathcal{A})/J \cong A/r$. Hence $\ker(\phi) \subseteq J$. Since $\phi(J^p) = r^p$ for all $p \geq 0$. We have $\phi(J^s) = r^s = 0$ for some $s > 0$. This completes the proof of the theorem. $\square$

Theorem 3.4 requires that the Artinian algebra A is splitting over its radical, i.e., the property (a) only without (b) as mentioned above, although the tenser algebra has to be replaced by the generalized path algebra. In case $r_A^2 = 0$, then the condition (b) is automatic provided (a) holds.

For an Artinian algebra A, setting $A_0 = A/r_A$ which is a semi-simple algebra with an A_0-A_0-bimodule structure on $M = r_A/r_A^2$. Then we can call the ordered pair $(A_0, r_A/r_A^2)$ a *generalized k-species* in the language of k-species discussed in [R]. The tensor algebra $T_{A_0}(M) = \oplus_{n=0}^{\infty} M^{\otimes_{A_0} n}$ is an associate algebra (not necessarily Artinian). The generalized path algebra of a quiver is naturally the tensor algebra of a generalized k-species. The properties of this path algebra is controlled by the bi-module structure of M. This algebra plays an important role in non-commutative geometry, which will be studied in the next paper.

The following example shows the difference between the generalized path algebra and the tensor algebra of the associated generalized k-species in the fact that ϕ does not exist with condition (a) only.

One also notes that the surjective algebra homomorphism $\pi : k(\Delta_A, \mathcal{A}) \to T_{A_0}(r_A/r_A^2)$ is an isomorphism if and only is $A_i(r_A/r_A^2)A_j$ is a free $A_i \otimes_k A_j$ -bimodule for all i, j. One natural question is, for a general π, whether there is a map $\psi : T_{A_0}(r_A/r_A^2) \to A$ such that $\psi \circ \pi = \phi$. The following example

gives an answer to this question. It also shows that one cannot expect to generalize [DK, Th. 8.5.2] to non-separable cases.

Example 3.5. Let $D = k[\alpha]$ with $k = \mathbb{F}_p(t)$ being the transcendental extension of the finite field with p elements and $\alpha = \sqrt[p]{t}$. Then D is a purely inseparable field extension of k. Now $B = D \otimes_k D$ is not semisimple and its radical r_B is nilpotent and generated by $x = \alpha \otimes_k 1 - 1 \otimes_k \alpha$ as a D-D-subbimodule of B. Let $A = D \oplus r_B$ as a k-vector space with multiplication defined by $(d, z)(d', z) = (dd', dz' + zd' + zz')$ with d, $d' \in D$ and z, $z' \in r_B$. Here dd' and zz' respectively the multiplications in D and r_B respectively by noting the D-D-bimodule structure on r_B. This makes A an associative algebra and $A/r_A = D$ is not separable. But A is splitting over its radical. A is a not a commutative algebra (unless $p = 2$) and $r_A/r_A^2 = r_B/r_B^2 \cong D$ as D-D-bimodules with the left and right actions of D coincide. Note the left and right actions of D on r_B are not the same unless $p = 2$. Hence the tensor algebra $T_D(r_A/r_A^2)$ is a commutative algebra. But A is not commutative. Hence there is no surjective k-algebra map $\psi : T_D(r_A/r_A^2) \to A$.

This example shows that under the assumption that A is splitting over its radical, the surjective map $\phi : k(\Delta_A, \mathcal{A}) \to A$ in Theorem 3.4 cannot be factored through $T_{A_0}(r_A/r_A^2)$.

As mentioned earlier that the separability condition on A_0 implies that A is splitting over its radical, by Wedderburn-Malcev theorem. There has been numerous generalizations of Wedderburn-Malcev theorem in the literature. Combining Theorem 3.3 and Theorem 3.4, the following gives a characterization of Artinian algebras that is splitting over its radical in terms of natural quivers and the associated generalized path algebras.

Corollary 3.6. *An Artinian k-algebra A is splitting over its radical if and only if A is of Gabriel-Type for generalized path algebras (cf. Def. 3.2), i.e., there is a surjective algebra homomorphism $\pi : k(\Delta_A, \mathcal{A}) \to A$ such that* $\ker(\pi) \subseteq J$*.*

We will see in Section 6 it will be important if the condition $\ker(\pi) \subset J$ in Corollary 3.6 is replaced by the stronger one, that is, $\ker(\pi) \subset J^2$. In this case, the ideal $I = \ker(\pi)$ is said to be *admissible*.

4. The relations among quivers arisen from an Artinian algebra to its generalized path algebra

In this section, we use the relation between the Ext-quiver and the natural quiver in Section 2 to study the associated normal generalized path algebras of Artinian algebras. Note that the definition of the Ext-quiver always requires that an Artinian algebra A is splitting over the field k, which we will assume in this section.

The natural quiver Δ_A of an Artinian algebra A is always finite, i.e. including finitely many vertices and finitely many arrows. In the sequel, we assume always that Δ_A is acyclic (i.e., Δ_A does not have oriented cycles of

length at least 1). Trivially, this is the sufficient and necessary condition under which the associated $k(\Delta_A, \mathcal{A})$ is Artinian.

By definition, acyclicity of Δ_A implies that the natural quiver of $k(\Delta_A, \mathcal{A})$ is just that of A, that is, $\Delta_{k(\Delta_A,\mathcal{A})} = \Delta_A$. Denote by m_{ij} and g_{ij} the arrow multiplicities respectively in the Ext-quivers Γ_A and $\Gamma_{k(\Delta_A,\mathcal{A})}$ respectively. Note that they should not be confused with each other. In general, m_{ij} and g_{ij} are quite different since the representation theories of A and $k(\Delta_A, \mathcal{A})$ are quite different (cf. Section 5).

For the radical J of $k(\Delta_A, \mathcal{A})$, $k(\Delta_A, \mathcal{A})/J \cong A/r_A = \bigoplus_{i\in\Delta_0} A_i$. By Theorem 2.2, the natural quiver $\Delta_{k(\Delta_A,\mathcal{A})}$ is a sub-quiver of the Ext-quiver $\Gamma_{k(\Delta_A,\mathcal{A})}$ and exactly,

$$t_{ij} = \lceil \frac{g_{ij}}{n_i n_j} \rceil. \tag{2}$$

Denote $I = (\Delta_A)_0$. A *complete set of non-isomorphic primitive orthogonal idempotents* of A is a set of primitive orthogonal idempotents $\{e_i : i \in I\}$ such that $Ae_i \ncong Ae_j$ as left A-modules for any $i \neq j$ in I and for each primitive idempotent e the module Ae is isomorphic to one of the modules Ae_i $(i \in I)$.

Let $\overline{e}_i = e_i + r_A$. Then, by [LC], $\{\overline{e}_i : i \in I\}$ is a complete set of non-isomorphic primitive orthogonal idempotents of A/r. Let $\tilde{e}_i \in k(\Delta_A, \mathcal{A})$ be the lift of $\bar{e}_i$ (we have assumed that $k(\Delta_A, \mathcal{A})$ is artinian). Then, $S_i \cong Ae_i/r_Ae_i = (A/r_A)\overline{e}_i \cong (k(\Delta_A, \mathcal{A})\tilde{e}_i)/(J\tilde{e}_i)$, as $i \in I$, give a list of all non-isomorphic irreducible modules for both A and $k(\Delta_A, \mathcal{A})$.

For $i \in I$, the identity 1_{A_i} of A_i can be decomposed into a sum of primitive idempotents, i.e. $1_{A_i} = e^i_{11} + \cdots + e^i_{n_in_i}$, and we can assume $e^i_{11} = e_i$.

Note that A is k-splitting. By ([ARS], Proposition III.1.14), we have

$$g_{ij} = \dim_k \operatorname{Ext}^1_{k(\Delta_A,\mathcal{A})}(S_i, S_j) = \dim_k(\tilde{e}^j_{kk} J/J^2 \tilde{e}^i_{ll})$$

for all k, l. Therefore,

$$\begin{aligned} \dim_k A_j J/J^2 A_i &= \dim_k 1_{A_j} J/J^2 1_{A_i} = \sum_{k=1}^{n_j}\sum_{l=1}^{n_i} \dim_k e^j_{kk} J/J^2 e^i_{ll} \\ &= n_i n_j \dim_k \operatorname{Ext}^1_{k(\Delta_A,\mathcal{A})}(S_i, S_j) = n_i n_j g_{ij}. \end{aligned}$$

Recall that the number of arrows from i to j in $\Delta_{k(\Delta_A,\mathcal{A})} = \Delta_A$ is t_{ij}. These t_{ij} arrows generate freely the A_j-A_i-bimodule $A_j J/J^2 A_i$. Thus,

$$\dim_k A_j J/J^2 A_i = \dim_k A_j \dim_k(J/J^2) \dim_k A_i = n_j^2 t_{ij} n_i^2.$$

Hence, $g_{ij} = n_i n_j t_{ij}$, that is, we obtain the following:

Proposition 4.1. *Let A be a k-splitting finite dimensional algebra with radical r, whose natural quiver Δ_A is acyclic. Write $A/r = \oplus_{i\in(\Delta_A)_0} A_i$ where A_i are simple algebra for all i with $n_i = \sqrt{\dim_k A_i}$. Then, the natural quiver of $k(\Delta_A, \mathcal{A})$ is the same with that of A and*

$$g_{ij} = n_i n_j t_{ij}$$

where g_{ij} is the number of arrows from i to j in $\Gamma_{k(\Delta_A,\mathcal{A})}$, t_{ij} is the number of arrows from i to j in Δ_A.

For the number m_{ij} of arrows from i to j in Γ_A, by Theorem 2.2, $t_{ij} = \lceil \frac{m_{ij}}{n_i n_j} \rceil$. Then, $\lceil \frac{m_{ij}}{n_i n_j} \rceil = \frac{g_{ij}}{n_i n_j}$, equivalently, we have

Corollary 4.2. $m_{ij} \leq g_{ij} < m_{ij} + n_i n_j$.

The set $\{P_i = Ae_i \mid i \in I\}$ is a complete set of representatives of the iso-class of indecomposable projective A-module. Then the basic algebra B of A is given by

$$B = \mathrm{End}_A(\coprod_{i \in I} P_i) \cong \bigoplus_{i,j \in I} \mathrm{Hom}_A(P_i, P_j) \cong \bigoplus_{i,j \in I} e_i A e_j.$$

Similarly the basic algebra of $k(\Delta_A, \mathcal{A})$ is

$$C = \mathrm{End}_{k(\Delta_A,\mathcal{A})}(\bigoplus_{i \in I} k(\Delta_A, \mathcal{A})\overline{e}_i) \cong \bigoplus_{i,\, j \in I} \overline{e}_i k(\Delta_A, \mathcal{A})\overline{e}_j.$$

The relationship between the two basic algebras B and C is given in [LC] when A is of Gabriel-type.

As we have said, $k(\Delta_A, \mathcal{A})$ is not Artinian when Δ_A has an oriented cycle. Hence, we cannot affirm whether C is Morita equivalent to $k(\Delta_A, \mathcal{A})$ in general. But, C is still decided uniquely by $k(\Delta_A, \mathcal{A})$. So, we call C the *basic algebra associated to* $k(\Delta_A, \mathcal{A})$.

As we have assumed Δ_A is acyclic in this section, C is Morita equivalent to $k(\Delta_A, \mathcal{A})$ since $k(\Delta_A, \mathcal{A})$ is Artinian. Hence, their Ext-quivers are the same, that is, $\Gamma_{k(\Delta_A,\mathcal{A})} = \Gamma_C$. And, since C is basic, $\Gamma_C = \Delta_C$.

To sum up, assuming that the Δ_A is acyclic, we get the following diagram:

$$\begin{array}{ccccccccc}
\Gamma_A & \overset{t_{ij}=\lceil \frac{m_{ij}}{n_i n_j} \rceil}{\supset} & \Delta_A & & = & & \Delta_{k(\Delta_A,\mathcal{A})} & \overset{t_{ij}=\frac{g_{ij}}{n_i n_j}}{\subset} & \Gamma_{k(\Delta_A,\mathcal{A})} \\
\| & & \cap & & & & \cap & & \| \\
\Gamma_B & = & \Delta_B & & \overset{m_{ij} \leq g_{ij} < m_{ij}+n_i n_j}{\subset} & & \Delta_C & = & \Gamma_C
\end{array}$$

where $\subset$, $\supset$ and $\cap$ mean the embeddings of the dense sub-quivers.

5. Diagram for non-splitting algebras

In Section 2 and 4, the Artinian algebra A is required to be splitting over the ground field k due to the definition of Ext-quiver. If A is not splitting over k, usually Ext-quiver and its representations have to be respectively replaced by the so-called *valued quiver* or k-*species*. On the other hand, the notion of the *diagram* of an Artinian algebra is introduced in [DK] in the case when A is not necessarily splitting over k. Now, we recall the definition of the diagram of an Artinian algebra A.

Let $P_1, P_2, \cdots, P_s$ be pairwise non-isomorphic principal indecomposable projective modules over an Artinian algebra A, corresponding to the simple

components $A_1, A_2, \cdots, A_s$ of the semisimple algebra $\overline{A} = A/r = \oplus_{i=1}^s A_i$. Write $R_i = rP_i$, then $S_i = P_i/R_i$ is the corresponding irreducible A-module. At the same time, $V_i = R_i/rR_i$ is a semisimple left A-module which has a direct sum decomposition $V_i \cong \oplus_{j=1}^s S_j^{\oplus h_{ij}}$ for some unique integers h_{ij}. Define the quiver $D_A = (D_0, D_1)$ by setting $D_0 = \{1, \ldots, s\}$ and the arrow set D_1 such that there are exactly h_{ij} many arrows from i to j for $i, j \in D_0$. This quiver D_A is called *the diagram of the algebra* A.

Observe that: (i) the projective cover of V_i is $P(R_i) \cong \oplus_{j=1}^s P_j^{\oplus h_{ij}}$ for each i; (ii) two Morita equivalent algebras have the same diagrams; (iii) the diagrams of A and A/r^2 coincide.

Proposition 5.1. *Let $S_1, S_2, \cdots, S_s$ be pairwise non-isomorphic irreducible modules over an Artinian algebra A. Then, in the diagram D_A of A, the number of arrows from the vertex i to the vertex j is*

$$h_{ij} = \dim_{D_j} \operatorname{Ext}_A(S_i, S_j)$$

where $D_j = \operatorname{End}_A(S_j)$ is a division algebra.

Proof. Applying the functor $\operatorname{Hom}_A(-, S_j)$ to the short exact sequence $0 \to R_i \to P_i \to S_i \to 0$, one gets the long exact sequence, which gives the isomorphisms

$$\operatorname{Ext}_A^p(R_i, S_j) \cong \operatorname{Ext}_A^{p+1}(S_i, S_j) \qquad (p \geq 0).$$

Since $R_i/rR_i = \oplus_{l=1}^s S_l^{\oplus h_{il}}$, we have $\operatorname{Hom}_A(R_i/rR_i, S_j) \cong D_j^{h_{ij}}$. Now the proposition follows from $\operatorname{Hom}_A(R_i/rR_i, S_j) \cong \operatorname{Hom}_A(R_i, S_j)$. $\square$

When A is k-splitting, each $D_j = k$, then $h_{ij} = \dim_k \operatorname{Ext}_A^1(S_i, S_j) = m_{ij}$. Hence, in this case, the Ext-quiver Γ_A is equal to the diagram D_A, whose relationship with Δ_A is given in Theorem 2.2. When A is not k-splitting, in the place of Ext-quiver Γ_A is a valued quiver with modulation. The following gives a relation of the diagram D_A with the valued quiver and associated modulation as well as with the natural quiver Δ_A for a general Artinian algebra A .

Using the earlier notations $A/r = A_1 \oplus \cdots \oplus A_s$ with $A_i \cong M_{n_i}(D_i)$ (the matrix algebra with entries in D_i). Let e_{pq}^i be the matrix basis element of A_i with 1 at (p, q) position and zero anywhere else. The irreducible left A_i-module $S_i \cong A_i e_{11}^i$ and irreducible right A_i-module $S_i^{op} \cong e_{11}^i A_i$. The A_j-A_i-module $A_j(r/r^2)A_i$ is semi-simple both as left A_j-module and as right A_i-module. Let D_l act on S_l from right and D_l act on S_l^{op} from left. For any A_j-A_i-bi-module M which is semisimple as A_j-module and A_i-module respectively, we have the following:

(a) $\operatorname{Hom}_{A_j}(S_j, M)$ is a D_j-A_i-bi-module isomorphic to $e_{11}^j M$ and the map $S_j \otimes_{D_j} \operatorname{Hom}_{A_j}(S_j, M) \to M$ defined by $x \otimes \phi = \phi(x)$ is an isomorphism of A_j-A_i-bi-modules;

(b) $\mathrm{Hom}_{A_i}(S_i^{op}, \mathrm{Hom}_{A_j}(S_j, M))$ is a D_j-D_i-bimodule isomorphic to $e_{11}^j M e_{11}^i$ and the map $\mathrm{Hom}_{A_i}(S_i^{op}, \mathrm{Hom}_{A_j}(S_j, M)) \otimes_{D_i} S_i^{op} \to \mathrm{Hom}_{A_j}(S_j, M)$ defined by $f \otimes v = f(v)$ is an isomorphism of D_j-A_i-bi-modules.

(c) The natural multiplication map $S_j \otimes_{D_j} e_{11}^j M e_{11}^i \otimes S_i^{op} \to M$ is an isomorphism of A_j-A_i-bi-module.

Taking $M = A_j(r/r^2)A_i$, we have an A_j-A_i-bi-module isomorphism

$$A_j(r/r^2)A_i \cong S_j \otimes_{D_j} e_{11}^j (r/r^2) e_{11}^i \otimes_{D_i} S_i^{op}. \tag{3}$$

Denote by $Arrow_{\Delta_A}(i,j)$ the k-linear space generated by the set of arrows from i to j in the natural quiver Δ_A. Then, $t_{ij} = \dim_k Arrow_{\Delta_A}(i,j)$ is the minimal number of generator of $A_j(r/r^2)A_i$ as A_j-A_i-module. Therefore there is a surjective homomorphism of A_j-A_i-bimodules

$$A_j \otimes_k Arrow_{\Delta_A}(i,j) \otimes_k A_i \twoheadrightarrow A_j(r/r^2)A_i.$$

In particular, by taking $M = A_j(r/r^2)A_i$, we have a surjective A_j-A_i-bimodule homomorphism

$$A_j \otimes_k Arrow_{\Delta_A}(i,j) \otimes_k A_i \twoheadrightarrow S_j \otimes_{D_j} e_{11}^j (r/r^2) e_{11}^i \otimes_{D_i} S_i^{op}. \tag{4}$$

Since $A_i \cong S_i \otimes_{D_i} S_i^{op}$ as A_i-A_i-bimodule, by applying the exactor functors $\mathrm{Hom}_{A_j}(S_j, -)$ and $\mathrm{Hom}_{A_i}(S_i^{op}, -)$ consecutively we get a surjective map of D_j-D_i-bimodules

$$S_j^{op} \otimes_k Arrow_{\Delta_A}(i,j) \otimes_k S_i \twoheadrightarrow e_{11}^j (r/r^2) e_{11}^i. \tag{5}$$

In the definition of the diagram D_A of A, we have $V_i \cong (r/r^2)e_{11}^i$ as left A-modules. Then by (a) and (b) above, $h_{ij} = \dim_{D_j}(e_{11}^j (r/r^2) e_{11}^i)$. Note that $S_j \otimes_{D_j} e_{11}^j (r/r^2) e_{11}^i \otimes_{D_i} S_i^{op}$ can be generated by h_{ij} many generators as an A_j-A_i-bimodule. Then (4) implies that $t_{ij} \leq h_{ij}$.

Since A is Artinian, we can compare the D_j-dimensions from (5) to get

$$h_{ij} \leq n_i n_j \dim_k(D_i) t_{ij}.$$

In conclusion, we have

Lemma 5.2. *For an Artinian k-algebra A, we have*

$$t_{ij} \leq h_{ij} \leq (n_i n_j dim_k D_i) t_{ij}.$$

Similar to the setup in Section 4, let $\varepsilon_i \in A$ be idempotents that are inverse images of e_{11}^i such that $\{\varepsilon_1, \ldots, \varepsilon_s\}$ is a complete set of non-isomorphic primitive idempotents of A. Set $\varepsilon = \sum_{i=1}^s \varepsilon_i$. Then $B = \mathrm{End}_A(A\varepsilon, A\varepsilon) \cong \varepsilon A \varepsilon$ is the basic algebra of A with radical $r_B = \varepsilon r_A \varepsilon$. We know that the diagram D_B of B is the same as that of A. In this case $h_{ij} = \dim_{D_j} \varepsilon_j r_B / r_B^2 \varepsilon_i$. The natural quiver Δ_B of B has the number of arrows $|Arrow_{\Delta_B}(i,j)|$ from i to j equal to the minimal number of generators of $\varepsilon_j r_B / r_B^2 \varepsilon_i$ as D_j-D_i-bimodules. Note that $\varepsilon_j r_B / r_B^2 \varepsilon_i \cong e_{11}^j r_A / r_A^2 e_{11}^i$ as D_j-D_i-bimodules. Then

$S_j \otimes_{D_j} e^j_{11} r_A/r_A^2 e^2_{11} \otimes_{D_i} S_i^{op}$ can be generated by $|Arrow_{\Delta_B}(i,j)|$ many elements as an A_j-A_i-bimodule. We therefore get

$$|Arrow_{\Delta_A}(i,j)| \leq |Arrow_{\Delta_B}(i,j)|. \tag{6}$$

For the basic algebra B, the system $\{\varepsilon_j r_B/r_B^2 \varepsilon_i \mid i,j = 1,\dots,s\}$ together with $\{D_i \mid i = 1,\dots,s\}$ defines a k-species [R] . If all D_i are finite dimensional over k, then the system is a modulation of the valued quiver for the algebra A [DR].

Theorem 5.3. *For an Artinian algebra A with radical r_A, let $\{S_1,\cdots,S_s\}$ be the complete set of all non-isomorphic irreducible A-modules and $D_i = \mathrm{End}_A(S_i)$ for any $i = 1,\cdots,s$. Let B be the basic algebra of A with radical r_B such that $B/r_B = \prod_{i=1}^s D_i$. Let t_{ij}^A (resp. t_{ij}^B) and h_{ij}^A (resp. h_{ij}^B) be the numbers of arrows from i to j in the natural quiver Δ_A (resp. Δ_B) and the diagram D_A (resp. D_B) respectively. For any $i,j = 1,\cdots,s$, we have*

(i) $t_{ij}^A \leq \lceil \frac{t_{ij}^B}{n_i n_j} \rceil \leq t_{ij}^B$;

(ii) $t_{ij}^A \leq \lceil \frac{h_{ij}^A}{n_i n_j} \rceil \leq t_{ij}^A \dim_k D_i$.

Proof. It follows from Proposition 5.1 that $h_{ij}^A = h_{ij}^B$. If M is a D_j-D_i-bimodule which can be generated by m many elements as a D_j-D_i-bimodule, then there is a surjective map $(D_j \otimes_k D_i)^{\oplus m} \to M$ as D_j-D_i-bimodules. Thus we have a surjective map $S_j \otimes_{D_j} (D_j \otimes_k D_i)^{\oplus m} \otimes_{D_i} S_i^{op} \to S_j \otimes_{D_j} M \otimes_{D_i} S_i^{op}$ of A_j-A_i-bimodules. Let $q = \lceil \frac{m}{n_j n_i} \rceil$. Hence we have a surjective map $(S_j \otimes_k S_i^{op})^{\oplus q n_j n_i} \to (S_j \otimes_k S_i^{op})^{\oplus m} \cong S_j \otimes_{D_j} (D_j \otimes_k D_i)^{\oplus m} \otimes_{D_i} S_i^{op}$ as A_j-A_i-bimodules. Since $(S_j \otimes_k S_i^{op})^{\oplus n_j n_i} \cong S_j^{\oplus n_j} \otimes_k (S_i^{op})^{\oplus n_i}$ as A_j-A_i-bimodules, which can be generated by one element as an A_j-A_i-bimodule, $(S_j \otimes_k S_i^{op})^{\oplus q n_j n_i}$ can be generated by q elements as A_j-A_i-bimodule. Therefore $S_j \otimes_{D_j} M \otimes_{D_i} S_i^{op}$ can be generated by q elements as an A_j-A_i-module.

Applying the above argument to $M = e_{11}^j (r_A/r_A^2) e_{11}^i$, where M can be generated by t_{ij}^B many elements as a D_j-D_i-bimodule, we conclude that $A_j(r_A/r_A^2)A_i$ can be generated by $\lceil \frac{t_{ij}^B}{n_j n_i} \rceil$ many elements as A_j-A_i-bimodules. This shows that $t_{ij}^A \leq \lceil \frac{t_{ij}^B}{n_j n_i} \rceil \leq t_{ij}^B$, where the first inequality follows from the fact that t_{ij}^A is the minimum.

To show (ii), we note that $t_{ij}^B \leq h_{ij}^B = h_{ij}^A \leq n_i n_j \dim_k D_i t_{ij}^A$ by Lemma 5.2. By (i), we have $t_{ij}^A \leq \lceil \frac{t_{ij}^B}{n_i n_j} \rceil \leq \lceil \frac{h_{ij}^A}{n_i n_j} \rceil \leq \lceil \frac{n_i n_j \dim_k D_i t_{ij}^A}{n_i n_j} \rceil = t_{ij}^A \dim_k D_i$. □

The relation (ii) implies that t_{ij} is never larger than h_{ij}, and $t_{ij} \neq 0$ if and only if $h_{ij} \neq 0$ for any $i,j \in (\Delta_A)_0$, which gives the relation between the natural quiver and the diagram.

The relation (i) gives the relation between the natural quiver and the Ext-quiver of an Artinian algebra over an arbitrary field k. Note that the

formula $t_{ij} = \lceil \frac{m_{ij}}{n_i n_j} \rceil$ in Theorem 2.2 holds only in the case that A is k-splitting, since the diagram D_A and the ext-quiver Γ_A are the same when A is splitting over k.

In general, when A is basic (i.e, $A_i = D_i$ is a division ring) without the splitting condition over k, the natural quiver Δ_A will not be the same as either Γ_A nor D_A. In this case, the discussion before Theorem 5.3 relates Δ_A and D_A by the species $\{D_i, \varepsilon_j r_A / r_A^2 \varepsilon_i\}$ [R] as follows.

(i) $h_{ij} = \dim_{D_j}(A_j r_A / r_A^2 A_i)$ and t_{ij} is minimal number of generators of $A_j r_A / r_A^2 A_i$ as A_j-A_i-bimodule;
(ii) if in addition that A is splitting over k, then $h_{ij} = t_{ij} = m_{ij}$, which means $D_A = \Delta_A = \Gamma_A$.

Example 5.4. Let E/k be a field extension. Consider the algebra $A = \begin{bmatrix} E & M \\ 0 & E \end{bmatrix}$ is a basic algebra over k for any E-E-bimodule M. With different choices of M one can have either $h_{ij} = t_{ij}$ (taking $M = E$) or h_{ij} and t_{ij} to be quite different (taking $M = E \otimes_k E$).

6. Hereditary algebras as generalized path algebras

We have mentioned the result in [DK] that any finite dimensional hereditary algebra A is isomorphic to the tensor algebra $T(A/r, r/r^2)$ if the quotient algebra A/r is separable for the radical r. As a comparison, in this section we will show that an Artinian hereditary algebra A is always isomorphic to $k(\Delta_A, \mathcal{A})$ if A is of Gabriel-type with admissible defining ideal.

It is proved in ([DK], Corollary 3.7.3) that the diagram D_A has no cycles if A is a finite dimensional hereditary algebra. In fact this is true for Artinian hereditary algebras.

Lemma 6.1. [HGK] *If a ring A is hereditary, then any nonzero homomorphism $\varphi : P_1 \to P_2$ of indecomposable projective A-modules is a monomorphism.*

Lemma 6.2. *If an Artinian algebra A is hereditary, then its diagram D_A has no cycles.*

Proof. For any vertices i and j, if there is an arrow σ from j to i in D_A, then there exists a non-zero homomorphism $f_\sigma : P_j \to P_i$ of projective covers irreducible modules S_j and S_i such that $f_\sigma(P_j) \subseteq rP_i$. By Lemma 6.1, f_σ is always a monomorphism, but not onto.

Suppose D_A has a cycle $\sigma_1\sigma_2\cdots\sigma_s$ with both tail and head at the vertex i. Then, $f = f_{\sigma_1} f_{\sigma_2} \cdots f_{\sigma_s}$ is a monomorphism from P_i to P_i since each f_{σ_i} is a monomorphism for any i. It is not isomorphic, i.e. $f(P_i) \subsetneqq P_i$. Moreover, it follows the infinite sequence:

$$P_i \supsetneqq f(P_i) \supsetneqq f^2(P_i) \supsetneqq \cdots \supsetneqq f^l(P_i) \supsetneqq \cdots .$$

Note that P_i is isomorphic to a left ideal of A, thus the above contradicts to the fact A is Artinian. $\square$

Proposition 6.3. *Let A be a hereditary Artinian algebra. Then the natural quiver Δ_A of A is finite and acyclic.*

Proof. Finiteness of Δ_A is trivial. According to the relation between Δ_A and D_A in Theorem 5.3, Δ_A is acyclic if and only if D_A is acyclic. By Lemma 6.2, D_A is acyclic. Hence, Δ_A is acyclic. □

By definition, normal generalized path algebras can be thought as a special class of tensor algebras, which are always hereditary due to [ARS]. The following main result in this section can be thought as a partial converse of this statement.

Proposition 6.4. *Let $\pi : B \to A$ be a surjective homomorphism of two hereditary algebras A and B such that $I = \ker(\pi) \subseteq \mathrm{rad}(B)^2$. If B is Artinian, then $I = 0$ and π is an isomorphism.*

Proof. Let $r_B = \mathrm{rad}(B)$. Since B is Artinian, then $r_B^n = 0$ for some n. We have $A \cong B/I$ and $r_B^s \subset I \subset r_B^2$ for some positive integer s. It is enough to prove that $I = 0$.

Let $R = r_B/I$. By induction on k, we will prove that for any $k \geq 0$,

$$R^k/R^{k+1} \cong r_B^k/r_B^{k+1} \tag{7}$$

as A-modules. Here the A-module structure of r_B^k/r_B^{k+1} can be induced naturally from the B-module structure of r_B^k/r_B^{k+1} since $I(r_B^k/r_B^{k+1}) = 0$.

When $k = 1$, we have $R/R^2 = (r_B/I)/(r_B/I)^2 = (r_B/I)/(r_B^2/I) \cong r_B/r_B^2$ since $I \subset r_B^2$.

Suppose that (7) holds for $k-1 \geq 0$, that is, $R^{k-1}/R^k \cong r_B^{k-1}/r_B^k$ as A-modules. For the case of k, we discuss as follows.

We first note that for any projective B-module Q, $Q/IQ = A \otimes_B Q$ is a projective A-module. In particular, for any A-module M, if $P_B(M)$ is projective cover of M as B-module, then $A \otimes_B P_B(M)$ is the projective cover of M/IM as A-module. Now let $\bar{P} = P_A(R^{k-1}/R^k)$ be the projective covers of R^{k-1}/R^k as A-modules, and let $Q = P_B(r_B^{k-1}/r_B^k)$ be the projective cover of r_B^{k-1}/r_B^k as B-module. Thus by induction assumption, $r_B^{k-1}/r_B^k \cong R^{k-1}/R^k$ as A-module. Then we have $\bar{P} \cong Q/IQ$ as A-modules (by [Lam] (pp. 363-364)). Let $\pi : Q \to Q/IQ$ being the quotient map, we have

$$\begin{aligned} R\bar{P} &\cong \pi(r_B)\pi(Q) = \pi(r_B Q) \cong r_B Q/IQ, \\ R^2\bar{P} &\cong \pi(r_B^2)\pi(Q) = \pi(r_B^2 Q) \cong r_B^2 Q/IQ. \end{aligned}$$

Since $R = r_B/I \cong \mathrm{rad}\,A$, we have $R^k = RR^{k-1} \cong (\mathrm{rad}\,A)R^{k-1} = \mathrm{rad}\,R^{k-1}$, thus $\bar{P} = P_A(R^{k-1}/R^k) = P_A(R^{k-1}/\mathrm{rad}\,R^{k-1}) = P_A(R^{k-1})$. Similarly, $r_B^k = \mathrm{rad}\,r_B^{k-1}$ and then, $Q = P_B(r_B^{k-1}/r_B^k) = P_B(r_B^{k-1})$. Here we are using the condition $I \subseteq r_B^2$.

Since A is hereditary, R is projective as A-module and we have $P_A(R^{k-1}) = R^{k-1}$. Similarly $P(r_B^{k-1}) = r_B^{k-1}$ since B is hereditary. Furthermore, $R^k \cong$

$R\bar{P}$, $R^{k+1} \cong R^2\bar{P}$ as A-modules and $r_B^k = r_BQ$ and $r_B^{k+1} = r_B^2Q$ as B-modules. Therefore we have the following isomorphisms of A-modules.

$$R^k/R^{k+1} = (R\bar{P})/(R^2\bar{P}) \cong (r_BQ/IQ)/(r_B^2Q/IQ) \cong r_B^k/r_B^{k+1}.$$

If one tracks the isomorphism, one would find that the above isomorphism is actually induced from the quotient map $\pi : B \to A$ since $\bar{P}$ and Q are subsets of A and B respectively (both algebras are hereditary).

Since B is Artinian, we have $r_B^m = 0$ for some m. If $I \neq 0$, there exists $p \geq 1$ minimal such that $I \not\subseteq r_B^p$. Then take $x \in I \backslash r_B^p$. It means $x \in r_B^{p-1}$ and $\pi(x) = 0 \in R^{p-1}$. But $\pi : r_B^{p-1}/r_B^p \to R^{p-1}/R^p$ is an isomorphism. Hence, the image of x in r_B^{p-1}/r_B^p has to be zero, i.e., $x \in r_B^p$ which contradicts the choice of x. Thus we must have $I = 0$. This proves the result. □

Theorem 6.5. *Let A be a hereditary Artinian algebra splitting over radical such that the surjective homomorphism $\pi : k(\Delta_A, \mathcal{A}) \to A$ in Theorem 3.4 possesses the kernel $ker(\pi) \subseteq J^2$. Then, π is an isomorphism.*

Proof. By Proposition 6.3, Δ_A is finite and acyclic. Then, $k(\Delta_A, \mathcal{A})$ is finite-dimensional and $\operatorname{rad} k(\Delta_A, \mathcal{A}) = J$ the ideal generated by all $\mathcal{A}$-paths of length 1. Thus, the condition in Proposition 6.4 is satisfied for the natural homomorphism from $k(\Delta_A, \mathcal{A})$ to A. It follows that $A \cong k(\Delta_A, \mathcal{A})$. □

In Theorem 6.5, A/r_A is not usually a separable algebra. Hence, this result can be thought as an improvement of [DK, Theorem 8.5.4].

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